ON MINIMAL COLLECTIONS OF INDEXES

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We denote $s = \left\lfloor \frac{n}{2} \right\rfloor$, $l = \left\lceil \frac{n+1}{2} \right\rceil$, $M_n = C_n^s = C_n^l$; indexes built for the case of $n$ columns (i.e., ordered subsets of the set $\{1, 2, ... , n\}$) will be called $n$-indexes.

The length of an index $a$ is designated by $\|a\|$. If $0 \leq i \leq \|a\|$, then $S_i(a)$ will denote the set consisting of the first $i$ elements of the index $a$. Any set that has exactly $i$ elements is said to be an $i$-element set. The cardinality of a set $A$ will be designated by $|A|$

Our aim is to build a collection of $n$-indexes $U$ such that for any subset $A \subseteq \{1, 2, ... , n\}$ there exists an index $a \in U$ satisfying the condition $S_\|A\|(a) = A$. For any $i \in \{0, 1, 2, ... , n\}$ every $i$-element subset of $\{1, 2, ... , n\}$ must agree with the beginning of at least one index, so the number of the indexes cannot be less than $C_i^n$, which is the number of different $i$-element subsets. In particular, if we consider the $s$-element subsets of $\{1, 2, ... , n\}$, we obtain that the number of the indexes is at least $M_n = C_s^n$.

At least $C_s^{n+1}$ of these $M_n$ (or more) indexes of length $s$ should be extended by one element. After that at least $C_s^{n+2}$ of the extended indexes should be extended by one more element, etc. Finally, one ($C_n^n = 1$) of the indexes already having length $n - 1$ should be extended by one more element. We obtain that the total length of the indexes cannot be less than $L_n = sC_s^n + C_s^{n+1} + C_s^{n+2} + ... + C_n^n$.

Recall that for any $i \in \{1, 2, ... , n\}$ the formula $C_{i+1}^n = C_i^n + C_i^{n-i}$ is valid. Next, for $i \in \{0, 1, 2, ... , n\}$ we have $C_i^n = C_{n-i}^n$. Note also that $\sum_{i=0}^{n} C_i^n = 2^n$. Then we obtain:

a) if $n = 2s + 1$, then $L_n = sC_s^n + \frac{1}{2} \cdot 2^n = sM_n + 2^{n-1}$;

b) if $n = 2s$, then $L_n = sC_s^n + \frac{1}{2}(2^n - C_s^n) = (s - \frac{1}{2})C_s^n + 2^{n-1} = (s - \frac{1}{2})M_n + 2^{n-1}$.

In both cases we have the equality $L_n = \frac{(n-1)M_n}{2} + 2^{n-1}$.

A collection $U$ of $n$-indexes is said to be minimal if:

1) the collection satisfies the conditions of the original problem, i.e., for any subset $A \subseteq \{1, 2, ... , n\}$ there exists an index $a \in U$ such that $S_\|A\|(a) = A$;

2) the collection has exactly $M_n$ indexes;

3) the total length of the indexes of the collection is $L_n$.

The above argument shows that a minimal collection of $n$-indexes (if such a collection exists) satisfies the following:
I. Every index of the collection has length at least \( l \) (for an odd number \( n \) this follows from the equality \( C_n^s = C_n^l \)).

II. The collection has exactly \( C_n^{l+1} \) indexes whose length is at least \( l + 1 \); exactly \( C_n^{l+2} \) indexes whose length is at least \( l + 2 \); ... ; \( C_n^{n-1} \) indexes whose length is at least \( n - 1 \), and one \( (C_n^s = 1) \) index whose length is \( n \).

An \( n \)-index \( a \) of a minimal collection is called long if \( \|a\| \geq l + 1 \) (the total number of such indexes is \( D_n = C_n^{l+1} \)). All other indexes of the collection will be called short, their total number being \( K_n = M_n - D_n \).

A minimal collection \( U \) of \( n \)-indexes is said to be good if it has the property:

for any natural number \( i \leq n \) and any \( i \)-element set \( A \subseteq \{1, 2, \ldots, n\} \)

there exists an index \( a \in U \) such that \( S_i(a) = A \) and \( \|a\| \geq n - i \).

Note that it suffices to check this condition for all \( i < \frac{n}{2} \) (if \( i \geq \frac{n}{2} \), then it is immediate that \( \|a\| \geq i \) and so \( \|a\| + i \geq n \)). It is also clear that the condition (*) holds for \( i = 0 \).

Let \( a \) be some \( n \)-index (we assume \( \|a\| \geq \frac{n}{2} \)). Denote by \( F_{n+1}(a) \) the \((n + 1)\)-index which is obtained by inserting the element \( n + 1 \) into the index \( a \) (the element must be inserted so that it becomes the \( m \)-th element of the index \( F_{n+1}(a) \), where \( m = n + 1 - \|a\| \leq \|a\| + 1 \)).

**Theorem.** For any natural number \( n \) there exists a good collection of \( n \)-indexes.

**Proof.** The basis \((n = 1)\) is obvious.

The inductive step will be divided into two cases.

a) Suppose that we already have a good collection of \((2k - 1)\)-indexes, say, \( U \). Define the collection \( V \) of \( 2k \)-indexes by the formula \( V = U \cup V' \), where \( V' = \{F_{2k}(a) \mid a \in U\} \). Now we check that the collection \( V \) is minimal.

1) Let \( A \subseteq \{1, 2, \ldots, 2k\} \) and \( B = A \setminus \{2k\} \), where \( |B| = i \). By the inductive assumption there exists a \((2k - 1)\)-index \( a \in U \subset V \) with the properties \( S_i(a) = B \) and \( \|a\| \geq 2k - 1 - i \). If \( 2k \notin A \), then \( S_i(a) = A \). Now suppose that \( 2k \in A \). Then the \( 2k \)-index \( b = F_{2k}(a) \in V \) has the element \( 2k \) in the \( m \)-th place, where \( m = 2k - \|a\| \leq i + 1 \). Hence \( S_{i+1}(b) = A \). So the collection \( V \) satisfies the conditions of the original problem.

2) The number of the indexes in the collection \( V \) is

\[
|V| = |U| + |V'| = 2M_{2k-1} = 2C_{2k-1}^{k-1} = C_{2k-1}^{k-1} + C_{2k-1}^k = C_{2k}^k = M_{2k}.
\]
3) The total length of the indexes of the collection $U$ is $L_{2k-1}$, hence the total length of the indexes of the collection $V'$ equals $L_{2k-1} + |V'| = L_{2k-1} + M_{2k-1}$. Then for the total length of the indexes of the collection $V$ we obtain the expression

$$2L_{2k-1} + M_{2k-1} = 2 \left( \frac{(2k - 2)M_{2k-1}}{2} + 2^{2k-2} \right) + M_{2k-1} = (2k - 2)M_{2k-1} + 2^{2k-1} + M_{2k-1} = (2k - 1)M_{2k-1} + 2^{2k-1} = \frac{(2k - 1)M_{2k}}{2} + 2^{2k-1} = L_{2k},$$

as desired. Thus the collection $V$ is really minimal.

We show that the collection $V$ is good. Let us choose $i \in \{1, 2, \ldots, k - 1\}$ and some $i$-element set $A \subseteq \{1, 2, \ldots, 2k\}$. Consider two cases.

- Let $2k \in A$; denote $B = A \setminus \{2k\}$, then $|B| = i - 1$. By the inductive assumption there exists a $(2k - 1)$-index $a \in U$ such that $S_{i-1}(a) = B$ and $\|a\| \geq 2k - i$. Then the $2k$-index $b = F_{2k}(a) \in V$ is obtained from $a$ by inserting the element $2k$ into the $m$-th place, where $m = 2k - \|a\| \leq i$. It means that $\|b\| \geq 2k - i$ and $S_{i}(b) = A$, i.e., the index $b$ guarantees the validity of the condition (*) for the collection $V$.

- Let $2k \notin A$. Then by the inductive assumption there exists a $(2k - 1)$-index $a \in U$ with $S_{i}(a) = A$ and $\|a\| \geq 2k - 1 - i$. If $\|a\| \geq 2k - i$, then the $2k$-index $a \in V$ guarantees the validity of the condition (*) for the collection $V$. And if $\|a\| = 2k - 1 - i$, then the $2k$-index $b = F_{2k}(a) \in V$ differs from $a$ only by the element $2k$ inserted into the $(i + 1)$-th place. Hence $\|b\| = 2k - i$ and $S_{i}(b) = A$. Thus the collection $V$ satisfies the condition (*).

b) Suppose that we already have a good collection of $2k$-indexes, say, $U$. Define the collection $V$ of $(2k + 1)$-indexes by the formula $V = V' \cup V''$, where $V'$ is the set of all long indexes of the collection $U$ and $V'' = \{F_{2k+1}(a) \mid a \in U\}$. Let us check that $V$ is minimal.

1) Let $A \subseteq \{1, 2, \ldots, 2k + 1\}$ and $B = A \setminus \{2k + 1\}$, where $|B| = i$. By the inductive assumption there exists a $2k$-index $a \in U$ with $S_{i}(a) = B$ and $\|a\| \geq 2k - i$. If $2k + 1 \notin A$ and $\|a\| \geq k + 1$, then $a \in V' \subset V$ and $S_{i}(a) = A$. If $2k + 1 \notin A$ and $\|a\| = k$, then the $(2k + 1)$-index $b = F_{2k+1}(a) \in V$ differs from $a$ only by the element $2k + 1$ added to the end, so $S_{i}(b) = A$.

Now suppose $2k + 1 \in A$. Then the $(2k + 1)$-index $b = F_{2k+1}(a) \in V$ has the element $2k + 1$ in the $m$-th place, where $m = 2k + 1 - \|a\| \leq i + 1$. Hence $S_{i+1}(b) = A$. Thus $V$ satisfies the conditions of the original problem.
2) The number of the indexes in the collection $V$ is

$$|V| = |V'| + |V''| = D_{2k} + M_{2k} = C_{2k}^{k+1} + C_{2k}^k = C_{2k+1}^k = M_{2k+1}.$$

3) The total length of the indexes of the collection $U$ is $L_{2k}$, so the total length of the indexes of the collection $V''$ is $L_{2k} + |V''| = L_{2k} + M_{2k}$. The total length of all short indexes of the collection $U$ equals $k \cdot K_{2k}$, and the total length of all long indexes equals $L_{2k} - k \cdot K_{2k}$. Then for the total length of the indexes of the collection $V$ we obtain the expression

$$2L_{2k} + M_{2k} - k \cdot K_{2k} = 2 \left( \frac{(2k - 1)M_{2k}}{2} + 2^{2k-1} \right) + M_{2k} - k(M_{2k} - D_{2k}) =$$

$$= (2k - 1)M_{2k} + 2^{2k} + M_{2k} - kM_{2k} + kD_{2k} = kM_{2k} + 2^{2k} + kD_{2k} =$$

$$= kC_{2k}^k + 2^{2k} + kC_{2k}^{k+1} = kC_{2k+1}^k + 2^{2k} = kM_{2k+1} + 2^{2k} = L_{2k+1}.$$

Hence the collection $V$ is minimal.

We show that the collection $V$ is good. Choose $i \in \{1, 2, \ldots, k\}$ and some $i$-element set $A \subseteq \{1, 2, \ldots, 2k+1\}$. Consider two cases.

- Let $2k + 1 \in A$; we designate $B = A \setminus \{2k + 1\}$, then $|B| = i - 1$. By the inductive assumption there exists a $2k$-index $a \in U$ such that $S_{i-1}(a) = B$ and $|a| \geq 2k + 1 - i$. Then the $(2k + 1)$-index $b = F_{2k+1}(a) \in V$ is obtained from $a$ by inserting the element $2k + 1$ into the $m$-th place, where $m = 2k + 1 - |a| \leq i$. It means that $|b| \geq 2k + 1 - i$ and $S_i(b) = A$, i.e., the index $b$ guarantees the validity of the condition (*) for the collection $V$.

- Let $2k + 1 \notin A$. Then by the inductive assumption there exists a $2k$-index $a \in U$ with $S_i(a) = A$ and $|a| \geq 2k - i$. If $|a| \geq 2k + 1 - i$, then the $2k$-index $a$ is long, i.e., the $(2k + 1)$-index $a \in V' \subset V$ guarantees the validity of the condition (*) for the collection $V$. And if $|a| = 2k - i$, then the $(2k + 1)$-index $b = F_{2k+1}(a) \in V$ differs from $a$ only by the element $2k + 1$ inserted into the $(i + 1)$-th place. Hence $|b| = 2k + 1 - i$ and $S_i(b) = A$. Thus the collection $V$ satisfies (*).
Good collections of $n$-indexes for small $n$

For $n = 1$:
- (1)

For $n = 2$:
- (2, 1)
- (1)

For $n = 3$:
- (3, 2, 1)
- (1, 3)
- (2, 1)

For $n = 4$:
- (4, 3, 2, 1)
- (1, 4, 3)
- (2, 4, 1)
- (3, 2, 1)
- (1, 3)
- (2, 1)

For $n = 5$:
- (5, 4, 3, 2, 1)
- (1, 5, 4, 3)
- (2, 5, 4, 1)
- (3, 5, 2, 1)
- (4, 3, 2, 1)
- (1, 3, 5)
- (2, 1, 5)
- (1, 4, 3)
- (2, 4, 1)
- (3, 2, 1)

For $n = 6$:
- (6, 5, 4, 3, 2, 1)
- (1, 6, 5, 4, 3)
- (2, 6, 5, 4, 1)
- (3, 6, 5, 2, 1)
- (4, 6, 3, 2, 1)
- (5, 4, 3, 2, 1)